OKOUNKOV BODY AND ITS VOLUME

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ABSTRACT

Let D be a big divisor on a projective variety X of dimension d. In this paper, we will investigate its Okounkov body, which is a compact convex set of $\mathbb{R}^d$ whose volume encodes the asymptotic behavior of hilbert series associated to the global sections ring $R(D) = \bigoplus_{m\in\mathbb{N}} H^0(X, \mathcal{O}_X(mD))$. This construction later will be generalized to a $\mathbb{N}^\rho$-graded linear series $W_\bullet \subseteq R(D_1, \ldots, D_\rho)$.

Keywords Hilbert series · Okounkov body · Graded linear series

1 Construction of the Okounkov body

In this section, we present the classical construction of the Okounkov body associated with a big divisor, refer to [LM09].

The Okounkov body is a compact convex set designed to study the asymptotic behavior of the complete linear series $H^0(X, \mathcal{O}_X(mD))$ as $m \to \infty$. Although we will see that Okounkov's construction works for incomplete linear series as well.

Definition 1.1. Given an irreducible variety X of dimension d. An **admissible flag** $Y_\bullet$ over X of length $l \leqslant d$ is defined as a flag of irreducible subvarieties

$$Y_\bullet : X = Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots \supseteq Y_{l-1} \supseteq Y_l,$$

where $\operatorname{codim}_X(Y_i) = i$ and each Y_i is non-singular at a general point of Y_l.

Notation 1.2. Throughout this paper, we will fix an admissible flag $Y_\bullet$ over an irreducible variety X of dimension d. Moreover, when we talk of a divisor, we refer to an integral Cartier divisor.

Definition 1.3 (Valuation attached to a flag)**.** Consider a big divisor D on X. A function

$$\begin{aligned}\nu = \nu_{Y_\bullet} = \nu_{Y_\bullet, D} : H^0\big(X, \mathcal{O}_X(D)\big) &\longrightarrow \mathbb{Z}^d \cup \{\infty\}\\ s &\longmapsto \nu(s) = \big(\nu_1(s), \ldots, \nu_d(s)\big)\end{aligned}$$

is called **valuation-like** if it satisfies

(i) $\nu_{Y_\bullet}(s) = \infty$ if and only if $s = 0$.

(ii) Ordering $\mathbb{Z}^d$ lexicographically, one has

$$\nu_{Y_\bullet}(s_1 + s_2) \geqslant \min\left\{\nu_{Y_\bullet}(s_1), \nu_{Y_\bullet}(s_2)\right\}$$

for any non-zero sections $s_1, s_2 \in H^0\left(X, \mathcal{O}_X(D)\right)$.

(iii) Given non-zero sections $s \in H^0\left(X, \mathcal{O}_X(D)\right)$ and $t \in H^0\left(X, \mathcal{O}_X(E)\right)$,

$$\nu_{Y_\bullet, D+E}(s \otimes t) = \nu_{Y_\bullet, D}(s) + \nu_{Y_\bullet, E}(t).$$

Such function would exists, the plan is to produce $\nu_i(s)$ inductively by restricting to each subvariety in the flag, and considering the order of vanishing along the next smallest. Specifically, given $0 \neq s \in H^0(X, \mathcal{O}_X(D))$ and set

$$\nu_1 = \nu_1(s) = \mathrm{ord}_{Y_1}(s).$$

Next, choose a local equation for s on an open neighborhood $U \subseteq X$ which determines a section

$$\tilde{s}_1 \in H^0(Y_1 \cap U, \mathcal{O}_{Y_1 \cap U}(D - \nu_1 Y_1))$$

that does not vanish identically along Y_1, and so we get by restricting a non-zero section

$$s_1 = \tilde{s}_{1|Y_1} \in H^0(Y_1, \mathcal{O}_{Y_1}(D - \nu_1 Y_1)).$$

Then take

$$\nu_2 = \nu_2(s) = \mathrm{ord}_{Y_2}(s_1).$$

Inductively, for $i \leqslant k$, one has constructed non-vanishing sections

$$s_i \in H^0\left(Y_i, \mathcal{O}_{Y_i}(D - \nu_1 Y_1 - \nu_2 Y_2 - \ldots - \nu_i Y_i)\right),$$

with $\nu_{i+1} = \mathrm{ord}_{Y_{i+1}}(s_i)$. Choosing a local equation for s on an open neighborhood $U \subseteq Y_k$ yields a section

$$\tilde{s}_{k+1} \in H^0(Y_{k+1} \cap U, \mathcal{O}_{U \cap Y_{k+1}}(D - \nu_1 Y_1 - \nu_2 Y_2 - \ldots - \nu_k Y_k) \otimes \mathcal{O}_{Y_{k+1}}(-\nu_{k+1} Y_{k+1}))$$

not vanishing along Y_{k+1}. Then take

$$s_{k+1} = \tilde{s}_{k+1|Y_{k+1}} \in H^0(Y_{k+1}, \mathcal{O}_{Y_{k+1}}(D - \nu_1 Y_1 - \nu_2 Y_2 - \ldots - \nu_{k+1} Y_{k+1})).$$

to continue the process. We then get the values $\nu(s) \in \mathbb{N}$ that do not depend on the choice of local equation $\tilde{s}_i$. It is immediate that properties (i) – (iii) are satisfied.

It follows from the valuation-like properties of $\nu_{Y_\bullet}$ that the valuations $\nu_{Y_\bullet}(s)$ along with their gradings form an additive semigroup in $\mathbb{N}^d \times \mathbb{N}$. We will make this precise:

Definition 1.4. The **graded semigroup** of D is the sub-semigroup

$$\Gamma(D) = \Gamma_{Y_\bullet}(D) = \{(\nu_{Y_\bullet}(s), m) \mid 0 \neq s \in H^0(X, \mathcal{O}_X(mD)), m \geqslant 0\}$$

of $\mathbb{N}^d \times \mathbb{N} = \mathbb{N}^{d+1}$. We also consider $\Gamma(D)$ as a subset of $\mathbb{Z}^{d+1} \subseteq \mathbb{R}^{d+1}$.

Definition 1.5. Writing $\Gamma = \Gamma(D)$ and denote $\Sigma(\Gamma) \subseteq \mathbb{R}^{d+1}$ for the closed convex cone with vertex at the origin spanned by Γ. The **Okounkov body** $\Delta(D)$ of D is then the base of this cone, that is

$$\Delta(D) = \Delta_{Y_\bullet}(D) = \Sigma(\Gamma) \cap (\mathbb{R}^d \times \{1\}).$$

Proposition 1.6. Let

$$\Gamma(D)_m = \mathrm{Im}(H^0(X, \mathcal{O}_X(mD) \setminus \{0\} \xrightarrow{\nu} \mathbb{Z}^d),$$

then we have another interpretation for Okounkov body

$$\Delta(D) = \mathrm{Conv}\left(\bigcup_{m \geqslant 1} \frac{1}{m}\Gamma(D)_m\right) \subseteq \mathbb{R}^d.$$

Example 1.7. On $X = \mathbb{P}^d$, let $Y_\bullet$ be the flag of linear spaces defined in homogeneous coordinates $T_0, \ldots, T_d$ by $Y_i = \{T_1 = \ldots = T_i = 0\}$ and take $L = \mathcal{O}_{\mathbb{P}^d}(1)$. The global sections $H^0(\mathbb{P}^d, \mathcal{O}(m))$ correspond to homogeneous polynomials of degree m in $d+1$ variables. Then $\nu_{Y_\bullet}$ is the lexicographic valuation determined on monomials by

$$\nu_{Y_\bullet}(T_0^{a_0} T_1^{a_1} \ldots T_d^{a_d}) = (a_1, \ldots, a_d).$$

Since

$$\Gamma(L)_m = \{(a_1, \ldots, a_d) \in \mathbb{N}^d \mid a_1 + \ldots + a_d = m - a_0,\ a_0 \in \mathbb{N}\},$$

the normalized points $\frac{1}{m}\Gamma(L)_m$ are the points lie in the standard simplex of $\mathbb{R}^d$

$$\left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_1 + \ldots + x_d \leqslant m\right\}.$$

Moreover, $\frac{1}{m}\Gamma(D)_m$ contains the standard basis by setting $a_0 = 0$. This shows that $\Delta(D)$ is exactly the standard simplex of $\mathbb{R}^d$.

Remark 1.8. For arbitrary divisors D it can happen that $\Delta(D) \subseteq \mathbb{R}^d$ has empty interior, in which $\Delta(D)$ isn't actually a convex body. For instance, take zero divisor $D = 0$, then $\Delta(D)$ consists of single point. However we will be almost exclusively interested in the case when D is big, and then $\mathrm{int}(\Delta(D))$ is indeed non-empty.

Lemma 1.9. Let $W \subseteq H^0(X, \mathcal{O}_X(D))$ be a subspace. Fix $a = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ and set

$$W_{\geqslant a} = \left\{s \in W \mid v_{Y_\bullet}(s) \geqslant a\right\}, \quad W_{> a} = \left\{s \in W \mid v_{Y_\bullet}(s) > a\right\}.$$

Then

$$\dim(W_{\geqslant a}/W_{>a}) \leqslant 1.$$

In particular, if W is finite dimensional then

$$\#\,\mathrm{Im}(W \setminus \{0\} \xrightarrow{\nu} \mathbb{Z}^d) = \dim W.$$

That is the number of valuation vectors arising from sections in W is equal to the dimension of W.

2 Volume of Okounkov body

Definition 2.1. Let L be a line bundle on X. The **support** of L consists of those non-negative powers of L that have a non-zero section:

$$\mathbb{N}(L) = \mathbb{N}(X, L) = \left\{ m \geqslant 0 \mid H^0(X, L^{\otimes m}) \neq 0 \right\}.$$

The semigroup $\mathbb{N}(X, D)$ of a divisor D is defined analogously with a line bundle $L = \mathcal{O}_X(D)$.

Definition 2.2 (Volume of a divisor)**.** Let L be a line bundle on X. The **volume** of L is defined to be the non-negative real number

$$\operatorname{vol}(L) = \operatorname{vol}_X(L) = \limsup_{m\to\infty} \frac{h^0(X, L^{\otimes m})}{m^d/n!}.$$

The semigroup $\mathbb{N}(X, D)$ and $\operatorname{vol}_X(D)$ of a divisor D is defined analogously with a line bundle $L = \mathcal{O}_X(D)$. Moreover, the divisor D on X is called **big** if there is a constant $C > 0$ such that

$$h^0(X, \mathcal{O}_X(mD)) \geqslant C \cdot m^d$$

for all sufficiently large $m \in \mathbb{N}(X, D)$. That is, the $\limsup$ above is in fact a limit

$$\operatorname{vol}_X(D) = \lim_{m\to\infty} \frac{h^0(X, \mathcal{O}(mD))}{m^d/d!}. \tag{1}$$

Definition 2.3. Given any semigroup $\Gamma \subseteq \mathbb{N}^{d+1}$, set

$$\Sigma = \Sigma(\Gamma) = \text{closed convex cone } (\Gamma) \subseteq \mathbb{R}^{d+1},$$
$$\Delta = \Delta(\Gamma) = \Sigma \cap (\mathbb{R}^{d+1} \times \{1\}).$$

Moreover for $m \in \mathbb{N}$, put

$$\Gamma_m = \Gamma \cap (\mathbb{N}^d \times \{m\}),$$

which we view as a subset $\mathbb{N}^d$. We do not assume that Γ is finitely generated, but we will suppose that it satisfies three conditions

(a) $\Gamma_0 = \{0\} \in \mathbb{N}^d$.

(b) $\exists$ finitely many vectors $(v_i, 1)$ spanning a semigroup $B \subseteq \mathbb{N}^{d+1}$ such that $\Gamma \subseteq B$.

(c) Γ generates $\mathbb{Z}^{d+1}$ as a group.

Proposition 2.4. Assume that Γ satisfies above conditions. Then

$$\lim_{m\to\infty} \frac{\#\Gamma_m}{m^d} = \operatorname{vol}_{\mathbb{R}^d}(\Delta),$$

where $\operatorname{vol}_{\mathbb{R}^d}$ denotes the standard Euclidean volume on $\mathbb{R}^d$.

Proof. The number of integral lattice points inside $m\Delta$ can be regard as a polynomal with respect to $m \in \mathbb{N}$, called the Ehrhart polynomial. Its leading coefficient of degree d is the volume of Δ [BRS15, Lemma 3.19], that is

$$\lim_{m\to\infty} \frac{\#(m\Delta \cap \mathbb{Z}^d)}{m^d} = \operatorname{vol}(\Delta).$$

And since

$$\Gamma_m \subseteq m\Delta \cap \mathbb{Z}^d,$$

it follows that

$$\limsup_{m\to\infty} \frac{\#\Gamma_m}{m^d} \leqslant \mathrm{vol}_{\mathbb{R}^d}(\Delta).$$

For the reverse inequality, assume to begin with that Γ is finitely generated. Khovanskii [Kho92, Proposition 3] shows that in this case there exists a vector $\gamma \in \Gamma$ such that

$$(\Sigma + \gamma) \cap \mathbb{N}^{d+1} \subseteq \Gamma.$$

here one uses that Γ generates $\mathbb{Z}^{d+1}$ as a group. But

$$\lim_{m\to\infty} \frac{\#(\Sigma+\gamma)\cap(\mathbb{N}^d \times \{m\})}{m^d} = \mathrm{vol}_{\mathbb{R}^d}(\Delta),$$

and hence

$$\liminf_{m\to\infty} \frac{\#\Gamma_m}{m^d} \geqslant \mathrm{vol}_{\mathbb{R}^d}(\Gamma). \tag{2}$$

This proves the theorem when Γ is finitely generated.

In general, choose finitely generated sub-semigroups

$$\Gamma^1 \subseteq \Gamma^2 \subseteq \ldots \subseteq \Gamma,$$

each satisfying (a) – (b), in such a manner that $\cup\Gamma^i = \Gamma$. Then $\#\Gamma_m \geqslant \#(\Gamma^i)_m$ for all $m \in \mathbb{N}$. Writing $\Delta^i = \Delta(\Gamma^i)$, it follows by applying (2) to Γ^i that

$$\liminf_{m\to\infty} \frac{\#\Gamma_m}{m^d} \geqslant \mathrm{vol}_{\mathbb{R}^d}(\Delta^i)$$

for all i. But $\mathrm{vol}_{\mathbb{R}^d} \to \mathrm{vol}_{\mathbb{R}^d}(\Delta)$ and so (2) holds for Γ as well. □

Lemma 2.5. If D is any big divisor on X, then the graded semigroup

$$\Gamma = \Gamma_{Y_\bullet}(D) \subseteq \mathbb{N}^{d+1}$$

associated to D satisfies the three conditions (a) – (b).

Proof. See [LM09, Lemma 2.2]. □

Theorem 2.6. Let D be a big divisor on a projective variety X of dimension d. Then

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)) = \frac{1}{d!}\,\mathrm{vol}_X(D). \tag{3}$$

Proof. Let $\Gamma = \Gamma(D)$ be the graded semigroup of D with respect to $Y_\bullet$. Thanks to Lemma 2.5, we can apply Proposition 2.4 and hence

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)) = \lim_{m\to\infty} \frac{\#\Gamma(D)_m}{m^d}.$$

On the other hand, it follows from 1.9 that $\#\Gamma(D)_m = h^0(X, \mathcal{O}_X(mD))$. By the definition at (1), the limit on the right computes $\frac{1}{d!}\,\mathrm{vol}_X(D)$. □

Example 2.7. Continuing the Example 1.7, the fact that $\Delta(D)$ is the standard simplex of dimension d helps us compute directly the geometric volume of $\Delta(D)$

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(D)) = \text{volume of standard } d\text{-simplex} = \frac{1}{d!}.$$

For the right hand side of (3), we know that

$$h^0(\mathbb{P}^d, \mathcal{O}(m)) = \binom{m+d}{d} \sim \frac{m^d}{d!}.$$

Therefore $\mathrm{vol}_X(D) = 1$.

Proposition 2.8. Let D be a big divisor on X.

(i) For a fixed natural number $a > 0$,

$$\mathrm{vol}(aD) = a^d \,\mathrm{vol}(D).$$

(ii) Fix any divisor N on X and any $\epsilon > 0$. Then there exists an integer p_0 such that

$$\frac{1}{p^d}\big|\mathrm{vol}(pD - N) - \mathrm{vol}(pD)\big| < \epsilon$$

for every $p > p_0$.

Proof. See [Laz04, Proposition 2.3.35]. □

Definition 2.9. (i) A $\mathbb{Q}$-divisor D on X is an element of $\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We represent a D as a finite sum

$$D = \sum c_i A_i$$

where $c_i \in \mathbb{Q}$ and $A_i \in \mathrm{Div}(X)$. By clearing denominators, we can also write $D = cA$ for a single rational number c and integral divisor A.

(ii) D is called effective if $c_i \geqslant 0$ and A_i effective.

(iii) D is called big if there is a positive integer $m > 0$ such that mD is integral and big.

(iv) Two $\mathbb{Q}$-divisors D_1, D_2 are numerically equivalent, written

$$D_1 \equiv_{\text{num}} D_2$$

if $(D_1 \cdot C) = (D_2 \cdot C)$ for every curve $C \subseteq X$. We denote by $N^1(X)_{\mathbb{Q}}$ for $\mathbb{Q}$-vector space of numerical equivalence classes of $\mathbb{Q}$-divisors. One can show that there is an isomorphism

$$N^1(X)_{\mathbb{Q}} = N^1(X) \otimes \mathbb{Q}.$$

(v) D is called ample if $c_i \in \mathbb{Q}_+$ and A_i is an ample Cartier divisor. Equivalently, D is ample if there is a positive integer $r > 0$ such that $r \cdot D$ is integral and ample.

(vi) We call $\xi \in N^1(X)_{\mathbb{Q}}$ an effective (big, ample) class if the representative element is effective (big, ample).

Definition 2.10 (Volume of $\mathbb{Q}$-divisor)**.** One can define volume of D by taking $\limsup$ over m for which mD is integral. However it would be quicker to choose some $a \in \mathbb{N}(D)$ for which aD is integral, then set

$$\mathrm{vol}(D) = \frac{1}{a^d}\,\mathrm{vol}(aD).$$

It follows from Proposition 2.8 (i) that this is independent of the choice of a.

Remark 2.11. Lazarsfeld and Mustata [LM09, Proposition 4.1] showed that the construction of Okounkov body does not depend on the integral numerical equivalence class. Moreover, if we regard $\Delta(\_)$ as a function on $\mathrm{Div}(X)$, then $\Delta(\_)$ satisfies homogenity condition. That is, given a big divisor D on X and an integer $p > 0$, one has

$$\Delta(pD) = p\Delta(D).$$

Therefore the Okounkov body $\Delta(\xi)$ is well defined for any big rational class $\xi \in N^1(X)_{\mathbb{Q}}$ by setting

$$\Delta(\xi) = \frac{1}{p}\Delta(pD) \subseteq \mathbb{R}^d \tag{4}$$

where D is a big $\mathbb{Q}$-divisor representing ξ and $p > 0$ is an integer large enough so that pD integral.

Proposition 2.12. For any big class $\xi \in N^1(X)_{\mathbb{Q}}$, we have

$$\mathrm{vol}_{\mathbb{R}^d}(\Delta(\xi)) = \frac{1}{d!}\,\mathrm{vol}_X(\xi).$$

Proof. Choose a $\mathbb{Q}$-divisor D representing ξ and an integer p such that pD integral. From the definition of $\Delta(\xi)$,

$$\mathrm{vol}(\Delta(\xi)) = \frac{1}{p^d}\,\mathrm{vol}(\Delta(pD)).$$

Also by the definition of volume of $\mathbb{Q}$-divisor and Theorem 2.6,

$$\mathrm{vol}_X(\xi) = \frac{1}{p^d}\,\mathrm{vol}_X(pD) = \frac{d!}{p^d}\,\mathrm{vol}_X(\Delta(pD)).$$

The assertion now follows. □

Definition 2.13. Analogously, one can defines $\mathbb{R}$-divisor be a element of $\mathrm{Div}_{\mathbb{R}}(X) = \mathrm{Div}(X) \otimes \mathbb{R}$. Write D as a finite sum $\sum c_i A_i$ where $c_i \in \mathbb{R}$ and $A_i \in \mathrm{Div}(X)$. It is numerically trivial if and only if $\sum c_i(A_i \cdot C) = 0$ for every curve $C \subseteq X$. The resulting space of equivalence classes is denoted by $\mathbb{N}^1(X)_{\mathbb{R}}$. We also has an isomorphism

$$N^1(X)_{\mathbb{R}} = \mathbb{N}^1(X) \otimes \mathbb{R}.$$

Definition 2.14. The **big cone** $\mathrm{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the convex cone of all big $\mathbb{R}$-divisor classes on X. The **pseudoeffective cone** $\overline{\mathrm{Eff}}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by the classes of all effective $\mathbb{R}$-divisors.

Theorem 2.15. The big cone is the interior of the pseudoeffective cone and the pseudoeffective cone is the closure of the big cone:

$$\mathrm{Big}(X) = \mathrm{int}(\mathrm{Eff}(X)), \quad \mathrm{Eff}(X) = \overline{\mathrm{Big}(X)}.$$

Lemma 2.16. The pseudoeffective cone of X is pointed, i.e. if $0 \neq \xi \in \mathrm{Eff}(X)$ then $-\xi \in \mathrm{Eff}(X)$.

3 Volume associated to a $\mathbb{N}^d$-graded linear series

Fix divisors $D_1, \ldots, D_\rho$ on X whose classes form a $\mathbb{Z}$-basis of $N^1(X)$. We may further choose $\{D_i\}$ so that their classes are in $\mathrm{Eff}(X)$. The choice of the $\{D_i\}$ determines identifications

$$N^1(X) = \mathbb{Z}^\rho, \quad N^1(X)_\mathbb{R} = \mathbb{R}^\rho.$$

Observe that under this isomorphism, $\mathrm{Eff}(X)$ lies in $\mathbb{R}^\rho_{\geqslant 0}$. Given a vector $\vec{a} \in \mathbb{N}^\rho$, we write $\vec{a} \cdot \vec{D} = a_1 D_1 + \ldots + a_\rho D_\rho$ for $\vec{D} = (D_1, \ldots, D_\rho)$.

Definition 3.1. An $\mathbb{N}^\rho$**-graded linear series** $W_\bullet$ on X associated to $\vec{D}$ consists of finite dimensional subspaces

$$W_{\vec{a}} \subseteq H^0(X, \mathcal{O}_X(\vec{a} \cdot \vec{D})).$$

for each $\vec{a} = (a_1, \cdots, a_\rho) \in \mathbb{N}^\rho$ such that

(i) $W_{\vec{0}} = \mathbb{C}$.

(ii) $W_{\vec{a}_1} \cdot W_{\vec{a}_2} \subseteq W_{\vec{a}_1 + \vec{a}_2}$ for all $\vec{a}_1, \vec{a}_2 \in \mathbb{N}^\rho$.

The product in (ii) denotes the image of $W_{\vec{a_1}} \otimes W_{\vec{a_2}}$ under the homomorphism

$$H^0(X, \mathcal{O}_X(\vec{a_1} \cdot \vec{D})) \otimes H^0(X, \mathcal{O}_X(\vec{a_2} \cdot \vec{D})) \to H^0(X, \mathcal{O}_X((\vec{a_1} + \vec{a_2}) \cdot D)).$$

Thus, above conditions is equivalent to the condition that $R(W_\bullet) = \oplus W_{\vec{a}}$ be a graded $\mathbb{C}$-subalgebra of the section ring

$$R(\vec{D}) = \bigoplus_{\vec{a} \in \mathbb{N}^\rho} H^0(X, \mathcal{O}_X(\vec{a} \cdot \vec{D})).$$

We define the support $\mathrm{Supp}(W_\bullet) \subseteq \mathbb{R}^\rho$ of $W_\bullet$ as the closed convex cone spanned by all $\vec{a} \in \mathbb{N}^\rho$ such that $W_{\vec{a}} \neq 0$.

Definition 3.2. The $\mathbb{N}^d$-graded semigroup of $W_\bullet$ with respect to a flag $Y_\bullet$ is the additive sub-semigroup of $\mathbb{N}^d \times \mathbb{N}^\rho$ given by

$$\Gamma(W_\bullet) = \left\{ (\nu(s), \vec{a}) \mid 0 \neq s \in W_{\vec{a}} \right\}.$$

Now let $\Sigma(W_\bullet) \subseteq \mathbb{R}^d \times \mathbb{R}^\rho$ be the closed cone spanned by $\Gamma(W_\bullet)$ and set

$$\Delta(W_\bullet) = \Sigma(W_\bullet).$$

Definition 3.3. For an $\mathbb{N}^\rho$-graded linear series $W_\bullet$ on X and $\vec{a} \in \mathbb{N}^\rho$, we define the volume function $\mathrm{vol}_{W_\bullet} : \mathbb{N}^\rho \to \mathbb{R}_+$ of $W_\bullet$ as

$$\mathrm{vol}_{W_\bullet}(\vec{a}) = \limsup_{k\to\infty} \frac{\dim_{\mathbb{C}}(W_{k\cdot\vec{a}})}{k^d/d!}.$$

With the help of convex geometry and semigroup theory, Lazarsfeld and Mustata [LM09, Corollary 4.20] show that the formal properties of the global volume function persist in the multigraded setting under very mild hypotheses. Precisely the function $\vec{a} \mapsto \mathrm{vol}_{W_\bullet}(\vec{a})$ extends uniquely to a continuous function

$$\mathrm{vol}_{W_\bullet} : \mathrm{int}(\mathrm{Supp}(W_\bullet)) \to \mathbb{R}_+.$$

which is homogeneous, log-concave of degree d.

Definition 3.4. Let $a \in \mathbb{N}^\rho$. An $\mathbb{N}^\rho$-graded linear series $W_\bullet$ on X has **bounded support** with respect to $\vec{a}$ if

$$\mathrm{Supp}(W_\bullet) \cap \left\{ \vec{b} \mid \vec{a} \cdot \vec{b} = 1 \right\}$$

is bounded. The **Reeb cone** of $\mathbb{N}^\rho$-graded linear series $W_\bullet$ on X is

$$\mathcal{C} = \left\{ \vec{a} \in \mathbb{N}^\rho \mid \langle \vec{a}, \vec{b} \rangle, \forall \vec{b} \in \mathrm{Supp}(W_\bullet) \setminus \{0\} \right\}.$$

A vector $\vec{a} \in \mathcal{C}$ is called **Reeb vector field**. For such $(W_\bullet, \vec{a})$ where $\vec{a}$ is a Reeb vector field, we set

$$h^0(W_{m,\vec{a},\bullet}) = \sum_{\vec{b}\cdot\vec{a}=m} \dim(W_{\vec{b}})$$

for each $m \in \mathbb{N}$, it is a finite sum if $W_\bullet$ has bounded support. Finally, we define the volume of $W_\bullet$ as

$$\mathrm{vol}_{\vec{a}}(W_\bullet) = \limsup_{m\to\infty} \frac{h^0(W_{m,\vec{a},\bullet})}{m^{d+\rho-1}/(d+\rho-1)!}.$$

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